Splints of root systems for special Lie subalgebras

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Abstract

Splint is a decomposition of root system into union of root systems. Splint of root system for simple Lie algebra appears naturally in studies of (regular) embeddings of reductive subalgebras. Splint can be used to construct branching rules. We consider special embedding of Lie subalgebra to Lie algebra. We classify projections of algebra root systems using extended Dynkin diagrams and single out the conditions of splint appearance and coincidence of branching coefficients with weight multiplicities. While such a coincidence is not very common it is connected with Gelfand-Tsetlin basis.

1   Introduction

The notion of splint was introduced by David Richter in the paper [Richter, 2012]. Splint is the decomposition of root system into disjoint union of images of two or more embeddings of some other root systems. Embedding ϕ of a root system Δ_1 into a root system Δ is a bijective map of roots of Δ_1 to a (proper) subset of Δ that commutes with vector composition law in Δ_1 and Δ.

$$\phi : \Delta_1 \longrightarrow \Delta$$

$$\phi \circ (\alpha + \beta) = \phi \circ \alpha + \phi \circ \beta, \ \ \alpha, \beta \in \Delta_1$$

Note that the image $Im(\phi)$ is not required to inherit the root system properties except the addition rules equivalent to the addition rules in Δ_1 (for pre-images). If an embedding ϕ preserves the angles between the roots it is called "metric". Two embeddings ϕ_1 and ϕ_2 can splinter Δ when the latter can be presented as a disjoint union of images $Im(\phi_1)$ and $Im(\phi_2)$.

Why one would consider non angle-preserving maps of root systems? Additive properties of root system determine the structure of Verma module:

$$M^\mu = U(\mathfrak{g}) \underset{U(\mathfrak{b}_+)}{\otimes} D^\mu(\mathfrak{b}_+) = \left\{ (E^{-\alpha_1})^{n_1} \ldots (E^{-\alpha_s})^{n_s} |v_\mu\rangle \right\}_{\alpha_i \in \Delta^+}^{n_i = 0, 1, \ldots}$$

Here $\mathfrak{b}_+$ is the Borel subalgebra of $\mathfrak{g}$, D^μ its one-dimensional representation, $|v_\mu\rangle$ is the highest weight vector and $E^{-\alpha_j}$ are lowering operators that are in correspondence with positive roots $\alpha_j \in \Delta^+$. Weyl character formula expresses character of irreducible representation as a combination of characters of Verma modules:

$$\mathrm{ch}L^\mu = \frac{\sum_{w\in W} \varepsilon(w)e^{w(\mu+\rho)-\rho}}{\sum_{w\in W} \varepsilon(w)e^{w\rho-\rho}} = \sum_{w\in W} \varepsilon(w)\,\mathrm{ch}M^{w(\mu+\rho)-\rho}$$

Here the sum is over elements of the Weyl group and their actions $w \triangleright \mu = w(\mu + \rho) - \rho$ depend on angles between roots. But we can rewrite Weyl character formula representing Weyl group elements as the products of reflections $s_\alpha, \alpha \in S$ in hyperplanes orthogonal to simple roots: $w = s_{\alpha_1} \cdot s_{\alpha_2} \dots$. Reflections act on root system by permutations, so the composition of such an action with embedding ϕ is easily obtained. The orbit of Weyl groups action $w \triangleright \mu, w \in W$, can be constructed by subtractions of roots from the highest weight μ. Let's denote Dynkin labels of μ by $(\mu_1, \dots \mu_r)$. Then $\mu - \mu_1\alpha_1, \mu - \mu_2\alpha_2, \dots, \mu - \mu_r\alpha_r, \alpha_i \in S$, are the points of the orbit that can be obtained by elementary reflections $s_{\alpha_i}, \alpha_i \in S$. Next set of points that are obtained by two consecutive reflections $s_{\alpha_i} \cdot s_{\alpha_j} \triangleright \mu$ are obtained as the subtraction $\mu - \mu_i\alpha_i - \mu_j(s_{\alpha_i}\alpha_j)$. Continuing this way we construct the image of singular element $\Psi = \sum_{w\in W} \varepsilon(w)e^{w(\mu+\rho)-\rho}$ after the embedding ϕ [Laykhovsky and Nazarov, 2012]. The multiplicities in the character of Verma module are unchanged by the embedding since they are determined by additive properties of the roots. So we see that weight multiplicities of irreducible modules are also preserved by the embedding.

Root system of regular subalgebra is contained in the root system of algebra so splint is useful in computation of branching coefficients.

In the paper [Laykhovsky and Nazarov, 2012] it was proven that the existence of splint leads to the coincidence of branching coefficients with weight multiplicities under certain conditions. We denote Lie algebra by $\mathfrak{g}$ and consider it's subalgebra $\mathfrak{a}$. If $\mathfrak{a}$ is a regular subalgebra, its root system $\Delta_\mathfrak{a}$ is contained in $\Delta_\mathfrak{g}$. Branching coefficients $b_\nu^{(\mu)}$ appear in decomposition of irreducible representation $L_\mathfrak{g}^\mu$ of $\mathfrak{g}$ to the sum of irreducible representations of $\mathfrak{a}$:

$$L_\mathfrak{g}^\mu = \bigoplus_\nu b_\nu^{(\mu)} L_\mathfrak{a}^\nu \tag{1}$$

Assume that root system $\Delta_\mathfrak{g}$ splinters as $\Delta_\mathfrak{g} = \Delta_\mathfrak{a} \cup \phi(\Delta_\mathfrak{s})$, where ϕ is an embedding of root system $\Delta_\mathfrak{s}$ of some semisimple Lie algebra $\mathfrak{s}$. Then branching coefficients $b_\nu^{(\mu)}$ for the reduction $L_{\mathfrak{g}\downarrow\mathfrak{a}}^{(\mu)}$ coincide with weight multiplicities $m_\nu^{(\tilde\mu)}$ in $\mathfrak{s}$-representations provided certain technical condition holds [Laykhovsky and Nazarov, 2012]. Highest weight $\tilde\mu$ of $\mathfrak{s}$-representation is calculated from Dynkin labels of highest weight μ of $\mathfrak{g}$-module.

In the next section 2 we review the classification of splints for simple root systems and results for branching coefficients. Then we move to the study of special embeddings which is the main subject of the present paper. We consider special embeddings of Lie subalgebras into a Lie algebra. In this case root

system of the subalgebra is not contained in the root system of the algebra. So the original motivation for splints is not applicable. But we can consider the projection of root system of the algebra on the root space of the subalgebra. Such a projection is not a root system anymore, but it satisfies milder conditions (Section 3). It is possible to classify most projections using Dynkin diagrams augmented with multiplicities. We then define splint for such 'weak' root systems and state the conditions of its appearance and implications for the calculation of branching coefficients (Section 4).

Use of representation theory of the subalgebra allows us to classify all the splints for the projections of algebra root system. We also apply this method to metric splints and regular subalgebras and get a unified treatment. We obtain Gelfand-Tsetlin rules for regular and special embeddings this way.

In conclusion 4 we discuss the cases when the projection of the root system does not fall into the classification mentioned above.

2 Splints of root systems and regular subalgebras

The notion and classification of splints for simple root systems were introduced in the paper [Richter, 2012].

Consider a simple Lie algebra $\mathfrak{g}$ and its regular subalgebra $\mathfrak{a} \hookrightarrow \mathfrak{g}$ such that $\mathfrak{a}$ is a reductive subalgebra $\mathfrak{a} \subset \mathfrak{g}$ with correlated root spaces: $\mathfrak{h}_{\mathfrak{a}}^* \subset \mathfrak{h}_{\mathfrak{g}}^*$. Let $\mathfrak{a}^{\mathfrak{s}}$ be a semisimple summand of $\mathfrak{a}$, this means that $\mathfrak{a} = \mathfrak{a}^{\mathfrak{s}} \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1) \oplus \dots$. We shall consider $\mathfrak{a}^{\mathfrak{s}}$ to be a proper regular subalgebra and $\mathfrak{a}$ to be the maximal subalgebra with $\mathfrak{a}^{\mathfrak{s}}$ fixed that is the rank r of $\mathfrak{a}$ is equal to that of $\mathfrak{g}$.

Computation of branching coefficients relies on roots $\Delta_{\mathfrak{g}} \backslash \Delta_{\mathfrak{a}}$ [Lyakhovsky and Nazarov, 2011a] so splint is naturally connected to branching coefficients for regular subalgebra.

Only three types of splints are injective and thus are naturally connected to branching [Richter, 2012]:

type	Δ	$\Delta_{\mathfrak{a}}$	$\Delta_{\mathfrak{s}}$
(i)	G_2	A_2	A_2
	F_4	D_4	D_4
(ii)	$B_r (r \geq 2)$	D_r	$\oplus^r A_1$
(*)	$C_r (r \geq 3)$	$\oplus^r A_1$	D_r
(iii)	$A_r (r \geq 2)$	$A_{r-1} \oplus u(1)$	$\oplus^r A_1$
	B_2	$A_1 \oplus u(1)$	A_2

$$(2)$$

Each row in the table gives a splint $(\Delta_{\mathfrak{a}}, \Delta_{\mathfrak{s}})$ of the simple root system Δ. In the first two types both $\Delta_{\mathfrak{a}}$ and $\Delta_{\mathfrak{s}}$ are embedded metrically. Stems in the first type splints are equivalent and in the second are not. In the third type splints only $\Delta_{\mathfrak{a}}$ is embedded metrically. The summands $u(1)$ are added to keep $r_{\mathfrak{a}} = r$. This does not change the principle properties of branching but makes it possible to use the multiplicities of $\mathfrak{s}$ -modules without further projecting their weights.

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Note that in the case of C_r-series (marked with the star in table (13)) metrical embedding $\Delta_{D_r} \to \Delta_{C_r}$ does not lead to the appearance of regular subalgebra $\mathfrak{a}$ (See [Dynkin, 1952b]).

In the paper [Lyakhovsky and Nazarov, 2011b] it was shown that

Property 2.1. *There is a decomposition of a singular element of the algebra $\mathfrak{g}$ into singular elements of the subalgebra $\mathfrak{s}$:*

$$\Phi_{\mathfrak{g}}^{\mu} = \sum_{w \in W_{\mathfrak{a}}} \epsilon\left(w\right) w \left(e^{\mu + \rho_{\mathfrak{g}}} \phi\left(e^{-\widetilde{\mu}} \Psi_{\mathfrak{s}}^{\widetilde{\mu}}\right)\right), \tag{3}$$

where $\epsilon\left(w\right)$ is a determinant of the element w of the Weyl group $W_{\mathfrak{a}}$ of the subalgebra $\mathfrak{a}$. The action of Weyl group on the weights is extended to the algebra of formal exponents by the rule $w\left(e^{\nu}\right) = e^{w\nu}, w \in W$. Hence, the multiplicity $M_{(\mathfrak{s})\widetilde{\nu}}^{\widetilde{\mu}}$ of the weight $\widetilde{\nu}$ from the weight diagram of the algebra $\mathfrak{s}$ with the highest weight $\widetilde{\mu}$ defines the branching coefficient $b_{\nu}^{(\mu)}$ for the highest weight $\nu = (\mu - \phi\left(\widetilde{\mu} - \widetilde{\nu}\right))$:

$$b_{(\mu - \phi(\widetilde{\mu} - \widetilde{\nu}))}^{(\mu)} = M_{(\mathfrak{s})\widetilde{\nu}}^{\widetilde{\mu}}. \tag{4}$$

3 Special embeddings and projections of root system

The study of specialy embedded subalgebras traces back to fundamental papers by Eugene Dynkin [Dynkin, 1952b, Dynkin, 1952a] where he called such subalgebras "S-subalgebras" to distinguish from regular or "R-subalgebras" that have root systems obtained by dropping some roots of the algebra root system. Thus regular subalgebras are easy to construct by dropping nodes from extended Dynkin diagram of the algebra, while the case of special subalgebras is more difficult. Complete classification for exceptional Lie algebras was obtained recently in [Minchenko, 2006]. The algorithm of the special subalgebras construction is available in GAP package but still requires manual intervention [de Graaf, 2011].

Assume that Lie algebra $\mathfrak{g}$ is simple. Let's denote a subalgebra by $\mathfrak{a}$. We denote corresponding Cartan subalgebras by $\mathfrak{h}_{\mathfrak{g}}$ and $\mathfrak{h}_{\mathfrak{a}}$ and identify them with the dual spaces $\mathfrak{h}_{\mathfrak{g}}^*$, $\mathfrak{h}_{\mathfrak{a}}^*$ using Killing forms of $\mathfrak{g}$ and $\mathfrak{a}$.

To construct an embedding $\mathfrak{a} \to \mathfrak{g}$ consider some representation $L_{\mathfrak{a}}^{\nu}$ as a subspace of Lie algebra $\mathfrak{g}$. Then one needs to check that the generators of $\mathfrak{a}$ can be presented as linear combinations of generators of $\mathfrak{g}$. We can identify Cartan subalgebra $\mathfrak{h}_{\mathfrak{a}} \subset \mathfrak{a}$ with dual space $\mathfrak{h}_{\mathfrak{a}}^*$ using Killing form. Then the root system $\Delta_{\mathfrak{g}}$ of $\mathfrak{g}$ can be projected to $\mathfrak{h}_{\mathfrak{a}}^*$ using the expression of $\mathfrak{h}_{\mathfrak{a}}$-generators through $\mathfrak{h}_{\mathfrak{g}}$-generators.

This projection is described in the classical papers [Dynkin, 1952b, Dynkin, 1952a] by the following theorem:

Theorem 1. *If a representation $L_{\mathfrak{g}}^{\mu}$ of the algebra $\mathfrak{g}$ induces a representation $L_{\mathfrak{a}}^{\widetilde{\mu}}$ on the subalgebra $\mathfrak{a}$ then the weight system of $L_{\mathfrak{a}}^{\widetilde{\mu}}$ can be obtained from $L_{\mathfrak{g}}^{\mu}$ by an orthogonal projection of $\mathfrak{h}_{\mathfrak{g}}^*$ on $\mathfrak{h}_{\mathfrak{a}}^*$.*

[Dynkin, 1952a].

In relation to the adjoint representation of $\mathfrak{g}$ this theorem means that the projection of the root system $\Delta_\mathfrak{g}$ to $\mathfrak{h}_\mathfrak{a}^*$ is a weight system of some finite-dimensional but not irreducible representation of $\mathfrak{a}$. Moreover this reducible representation contains the adjoint representation of $\mathfrak{a}$. And if we denote such a projection by

$$\Delta' = \pi_\mathfrak{a}\left(\Delta_\mathfrak{g}\right), \tag{5}$$

then the roots of $\mathfrak{a}$ will be contained in Δ'. Thus the system Δ' can be a root system but it might not be reduced and might contain some vectors with multiplicities greater than one.

The following theorems [Dynkin, 1952b] also concern the properties of Δ'.

Theorem 2. *Every special subalgebra $\mathfrak{a}$ of a semisimple Lie algebra $\mathfrak{g}$ is integer, i.e. the projections of roots of $\mathfrak{g}$ to $\mathfrak{h}_\mathfrak{a}^*$ are linear combinations of simple roots of $\mathfrak{a}$ with integer coefficients [Dynkin, 1952b]*

Theorem 3. *If $\mathfrak{a}$ is a semisimple subalgebra of a semisimple Lie algebra $\mathfrak{g}$ and the generator e_α corresponding to the root α of $\mathfrak{a}$ is presented as a linear combination $e_\alpha = \sum_\beta e_\beta$ of $\mathfrak{g}$-generators corresponding to the roots β of $\mathfrak{g}$, then the roots β are projected into α, $\pi_\mathfrak{a}(\beta) = \alpha$. [Dynkin, 1972, Dynkin, 1952b].*

From these theorems we see that the root system $\Delta_\mathfrak{a}$ is metrically embedded (in the sense of splint) into the projection Δ'. Moreover, the multiplicities of some roots $\alpha \in \Delta_\mathfrak{a}$ in this projection Δ' are greater than one.

As an example we present a projection Δ' of the root system of D_4 to the root system of the special subalgebra A_2 (Fig. 1).

4 Splints for special embeddings

We want to classify all the cases when branching coefficients coincide with weight multiplicities of some other algebra representations. Such a coincidence is possible when the projection Δ' of $\mathfrak{g}$ root system admits a splint $\Delta' = \varphi_\mathfrak{a}(\Delta_\mathfrak{a}) \cup \varphi_\mathfrak{s}(\Delta_\mathfrak{s})$, where $\varphi_\mathfrak{a}$ and $\varphi_\mathfrak{s}$ are the embeddings of corresponding root systems. The embedding $\varphi_\mathfrak{a}$ is metric and trivial. Moreover, the projection of singular element $\pi_\mathfrak{a}\left(\Psi_\mathfrak{g}^\mu\right)$ should admit a decomposition into linear combination of singular elements of irreducible representations of $\mathfrak{s}$:

$$\pi_\mathfrak{a}\left(\Psi_\mathfrak{g}^\mu\right) = \sum_\nu \varkappa_\nu e^\nu \Psi_\mathfrak{s}^{\tilde{\mu}}, \tag{6}$$

with integer coefficients $\varkappa_\nu$, where the sum is over the set of weights ν that will be determined later. First we classify all the splints of the projection of a root system into a union of simple root systems for special embeddings, and then discuss the decomposition of singular elements.

Projection of the $\mathfrak{g}$ root system Δ' coincides with the projection of the weight diagram of the adjoint representation of algebra $\mathfrak{g}$ with the exception of zero
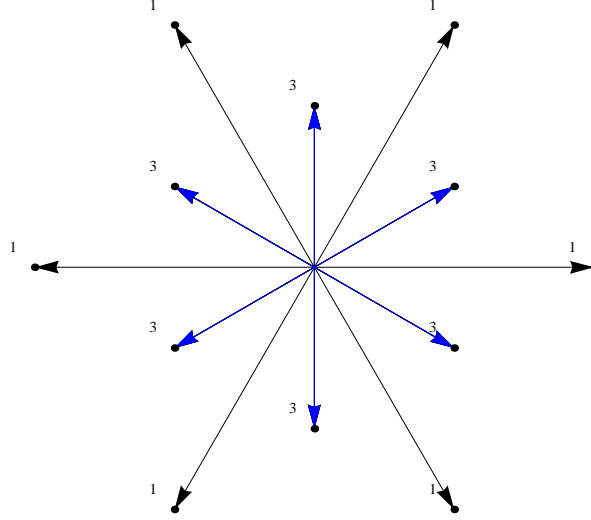
Figure 1: Projection of D_4 ($so(8)$)-root system onto the root space of the special subalgebra A_2 ($su(3)$). Note that this roots system is G_2 but with non-trivial multiplicities.

weight. The adjoint representation of $\mathfrak{g}$ contains adjoint representation of $\mathfrak{a}$ and can be decomposed as

$$\mathrm{ad}_{\mathfrak{g}} = \mathrm{ad}_{\mathfrak{a}} \oplus M_{\mathfrak{a}}^{\chi}, \tag{7}$$

where $M_{\mathfrak{a}}^{\chi}$ is not necessary irreducible representation of $\mathfrak{a}$ called "characteristic" in [Dynkin, 1952b].

The weight diagram of $M_{\mathfrak{a}}^{\chi}$ coincides with the root system $\Delta_{\mathfrak{s}}$ with the exception of zero weight. So we need to find all the representations for all simple Lie algebras $\mathfrak{a}$, such that their weight diagrams are root systems after the exclusion of zero weight. In order to do so we must note, that all the weights of $M_{\mathfrak{a}}^{\chi}$ should have length not less than length of shortest root of $\mathfrak{a}$, since $\mathfrak{a}$ is an integer subalgebra of $\mathfrak{g}$ 2. Moreover, $M_{\mathfrak{a}}^{\chi}$ should contain weights of at most two different lengths.

If the root system $\Delta_{\mathfrak{g}}$ contains roots $\alpha : \alpha \perp \beta, \ \forall \beta \in \Delta_{\mathfrak{a}}$, orthogonal to the root system of the subalgebra, one cannot proceed with the decomposition of singular element (6). The elements e^{ν} of $\pi_{\mathfrak{a}}\left(\Psi_{\mathfrak{g}}^{\mu}\right)$ should be augmented with the dimensions of representations of algebra $\mathfrak{a}_{\perp}$ spanned by generators corresponding to orthogonal roots [Lyakhovsky and Nazarov, 2011a]. Then there will be non-trivial multiplicities in the right hand side of (6) and no coincidence of branching coefficients with weight multiplicities of $\mathfrak{s}$-representations. One can write more cumbersome relation between multiplicities and branching coefficients, but it is out of scope for the present paper.

The multiplicity of zero weight in the adjoint representation is equal to the rank of the algebra. So if $\mathrm{rank}\mathfrak{g} - \mathrm{rank}\mathfrak{a} = 1$, the representation in question

$M_\mathfrak{a}^\chi$ must be multiplicity-free. If $\mathrm{rank}\,\mathfrak{g} - \mathrm{rank}\,\mathfrak{a} > 1$ only zero weight can have non-trivial multiplicities.

The simplest class of multiplicity-free representations is called "strongly multiplicity free" [Lehrer and Zhang, 2006] and consists of multiplicity free representations with weight systems admitting strict ordering $\nu_1 < \nu_2 \Leftrightarrow \nu_1 = \nu_2 + n\alpha$, where $n \in \mathbb{N}, \alpha \in \Delta_\mathfrak{a}^+$ and for all ν_1, ν_2 either $\nu_1 < \nu_2$ or $\nu_2 < \nu_1$.

The list of strongly multiplicity free representations consists of (first) fundamental representations for series A_r, B_r, C_r, exceptional Lie algebra G_2 (7-dimensional representation) and all A_1 representations.

Fundamental weights of the algebra A_r are shorter than its roots, but the projections of $\mathfrak{g}$ weights must be given by linear combinations of the roots of the special subalgebra $\mathfrak{a}$ with integer coefficients, so the first class of strongly multiplicity free representations does not produce such a projection.

Nevertheless, the union of diagrams of two fundamental representations of A_2 with the root system of A_2 produces weight diagram of G_2. This case corresponds to splint $\Delta_{G_2} = \varphi_1(\Delta_{A_2}) \cup \varphi_2(\Delta_{A_2})$ which is connected with regular subalgebra $A_2 \subset G_2$ (see Section 2).

First fundamental representation of B_r immediately gives us the splint $\Delta' = \pi_{B_r}\left(\Delta_{D_{r+1}}\right) = \Delta_{B_r} \cup \Delta_{A_1 + \cdots + A_1}$ corresponding to Gelfand-Tsetlin multiplicity-free branching for the special embedding $\mathfrak{so}(2r+1) \to \mathfrak{so}(2r+2)$.

The length of the first fundamental weight of C_r is less than the length of its short root, so this case can not correspond to an integer subalgebra.

If there are vectors of different length in the projection Δ' on the subalgebra A_1, such a projection is not multiplicity free, and $\Delta_\mathfrak{s}$ contains parallel roots. The simplest case is the special embedding $A_1 \to A_2$ with index 4 where $\Delta_\mathfrak{s}$ is the root system BC_1. Such systems do not correspond to semisimple Lie algebras, so we can not speak of a coincidence of branching coefficients for the reduction $\mathfrak{g} \downarrow \mathfrak{a}$ with weight multiplicities of $\mathfrak{s}$-representations.

The weight diagram of the seven-dimensional representation of G_2 together with G_2 root system form the projection of the root system B_3 which corresponds to special embedding $G_2 \to B_3$. Branching coefficients in this case coincide with weight multiplicities of representations of algebra $\mathfrak{s} = A_2$ with image of root system consisting of short roots of G_2.

The complete classification of multiplicity-free irreducible representations was obtained in [Howe, 1995] (see also [Stembridge, 2003]). It consists of minuscule and quasi-minuscule representations.

The weight μ is minuscule if $\langle \mu, \alpha^\vee \rangle \leq 1$ for all $\alpha \in \Delta^+$ and all the weights of the irreducible representation L^μ lie on the Weyl group orbit of μ. A weight is quasi-minuscule if $\langle \mu, \alpha^\vee \rangle \leq 2$ for all $\alpha \in \Delta^+$ and all non-zero weights lie on the single Weyl group orbit.

Minuscule representations are well-studied and very useful in computations of e.g. tensor products [Stembridge, 2003, Stembridge, 2001]. The minuscule representations are indexed by the weight lattice modulo the root lattice. There is a unique quasi-minuscule representation that is not minuscule for each simple Lie algebra. The multiplicity of the zero weight in quasi-minuscule representation is the number of short nodes of the Dynkin diagram.

List of minuscule representations consists of tensor powers of vector representation for the series A_r; spin representations for the series B_r; vector representations for C_r; vector and two half-spin for D_r; two 27-dimensional representations for E_6 and 56-dimensional representation of E_7.

Quasi-minuscule representations that are not minuscule are: adjoint representation of A_r, vector representation for B_r, $2r^2 - r - 1$-dimensional representation for C_r, adjoint representations for D_r, E_6, E_7, E_8, 26-dimensional representation for F_4 and 7-dimensional of G_2.

Classification of multiplicity-free irreducible highest weight modules L^μ in [Howe, 1995, Stembridge, 2003] consists of the following classes:

- (1) μ is minuscule,

- (2) μ is quasi-minuscule and Lie algebra has only one short simple root,

- (3) Lie algebra is $\mathfrak{sp}(6)$ and $\mu = \omega_1$, or

- (4) Lie algebra is $\mathfrak{sl}(n+1)$ and $\mu = m\omega_1$ or $\mu = m\omega_n$

We see that case (1) contains strongly multiplicity free modules of series A_r and exceptional Lie algebra G_2, class (2) contains strongly multiplicity free modules of B_r and C_r , strongly multiplicity free modules of A_1 are included in class (4).

We do not need to consider minuscule representations for simply-laced Lie algebras, since in this case length of the weights of representation $M_{\mathfrak{a}}^\chi$ is less than length of $\mathfrak{a}$ roots, that contradicts to $\mathfrak{a}$ being an integer subalgebra. For the series B_r we have already discussed minuscule spin representation, since it is strongly multiplicity free. The last case in (1) is the vector representation for C_r. It is $2r$-dimensional, but the length of its highest weight ω_1 is less than the length of the short root of algebra C_r. So it can not be a characteristic representation for the special subalgebra.

Case (2) consists of the series B_r and the exceptional root system G_2. The embedding $G_2 \rightarrow B_3$ was already discussed. The dimension of the adjoint representation for the series B_r is $2r^2 + r$ and of quasi-minuscule representation it is $2r + 1$, so the dimension of the algebra $\mathfrak{g}$ should be $(r+1)(2r+1)$ with the rank$\mathfrak{g} = r + 1$. The only solution is $\mathfrak{g} = D_{r+1}$ and we have already seen that this case corresponds to strongly multiplicity free module and Gelfand-Tsetlin basis.

In case (3) the diagram of the representation contains weights with the length smaller than the length of short root, so it cannot be a projection of $\Delta_{\mathfrak{g}}$ to an integer subalgebra.

We need to consider representations of class (4) only for $m = 1, 2$ since the root system $\Delta_{\mathfrak{s}}$ can have roots of at most two different lengths. In the case $m = 1$ we get strongly multiplicity free modules of A_r which were discussed above. And for algebra A_2 and $m = 2$ the weight diagram of representation $L_{A_2}^{2\omega_1}$ coincides with the root system of exceptional Lie algebra G_2. This case appears in the special embedding $A_2 \rightarrow B_3$. But for $A_r, r > 2$ there are no root systems that coincide with weight diagram of $L_{A_r}^{2\omega_1}$.

So far we've considered all the cases when $\mathrm{rank}\mathfrak{g} - \mathrm{rank}\mathfrak{a} = 1$. To complete our classification of splints we need the classification of all the representations where multiplicities of all non-zero weights are equal to one. Fortunately, such a classification was obtained in [Plotkin et al., 1998]. It consists of multiplicity-free representations and adjoint representations of algebras A_r, B_r, C_r, F_4 and G_2.

In order to have the embedding $\mathfrak{a} \to \mathfrak{g}$ such that the adjoint representation of $\mathfrak{g}$ is decomposed into two adjoint representations of $\mathfrak{a}$ (and possibly several trivial representations) we need to satisfy following conditions:

- Denote by $n_\mathfrak{g}$ the total number of roots in $\Delta_\mathfrak{g}$ and the projection Δ'. Then

$$n_\mathfrak{g} = 2n_\mathfrak{a}. \tag{8}$$

- Denote rank of $\mathfrak{g}$ by shorthand notation $r_\mathfrak{g}$. Then the dimension

$$\dim\mathfrak{g} = n_\mathfrak{g} + r_\mathfrak{g} \geq 2\dim\mathfrak{a} = 2n_\mathfrak{a} + 2r_\mathfrak{a} \tag{9}$$

For the adjoint representation of algebra A_r we see that the projection Δ' of the root system $\Delta_\mathfrak{g}$ should consist of roots of subalgebra $\mathfrak{a} = A_r$ with multiplicity 2. The total number of roots for A_r is $r(r+1)$, so Δ' and $\Delta_\mathfrak{g}$ contain $2r(r+1)$ roots, since there are no roots orthogonal to $\Delta_\mathfrak{a}$. There are two possible solutions for $\mathfrak{g}$: D_{r+1} and G_2 for $r = 2$. But the dimension of D_{r+1} is $(r+1)(2r+1)$ while the dimension of A_r is $r(r+2)$ and $2\dim A_r > \dim D_{r+1}$, so the adjoint representation of D_{r+1} cannot be decomposed as twice the adjoint representation of A_r. There is no corresponding embedding $A_r \to D_{r+1}$ and no splint. We see that conditions (8)(9) do not hold.

Let's consider the embedding $B_r \to \mathfrak{g}$ such that the adjoint representation of $\mathfrak{g}$ is decomposed into two adjoint representations of B_r. It is straightforward to check that there is no solutions for $\mathfrak{g}$ satisfying the (8)(9) for the series A, B, C, D and all exceptional simple Lie algebras.

Since the number of roots of root system C_r is the same as for B_r and we need to check the same cases, we see that there is no embedding for C_r too.

The adjoint representation of G_2 gives us the algebra D_4 as the solution of the constraints (8)(9). The embedding $G_2 \to D_4$ exists, since there are embeddings $G_2 \to B_3$ and $B_3 \to D_4$, but the decomposition of adjoint representation of D_4 is different: $\mathrm{ad}_{D_4} = \mathrm{ad}_{G_2} \oplus 2L_{G_2}^{\omega_1}$. So this case does not produce splint for the projection Δ'.

For the adjoint representation of F_4 there is no solution that satisfies conditions (8)(9).

The complete classification of splints for special embeddings $\Delta' = \pi_\mathfrak{a}(\Delta_\mathfrak{g}) \equiv \Delta_\mathfrak{a} \cup \Delta_\mathfrak{s}$ where $\mathfrak{g}, \mathfrak{a}$ are simple and $\mathfrak{s}$ is semisimple is:

- $B_n \to D_{n+1}$

- $G_2 \to B_3$

Note that in the splints for special embeddings there is no case where $\Delta_{\mathfrak{s}}$ is embedded non-metrically which is a direct result of our exhaustive classification of suitable characteristic representations of the subalgebra $\mathfrak{a}$.

Having obtained the complete classification of splints of the projections of the root systems for special embeddings, we need to check whether such splints lead to the coincidence of branching coefficients with weight multiplicities in modules of the algebra $\mathfrak{s}$.

Let's consider the first case - the embedding $B_n \to D_{n+1}$. The root system of D_{n+1} has exactly two simple roots that are different from the simple roots of B_n. In the notation of [Bourbaki, 2002] they are $\alpha_n = e_n - e_{n+1}$ and $\alpha_{n+1} = e_n + e_{n+1}$ while $\alpha_n^{B_n} = e_n$. This difference leads to the crucial difference in the Weyl groups of the algebras: while W_{B_n} is a semidirect product of the group of permutation $e_i \to e_j$ and the group of the change of sign $e_i \to (\pm 1)_i e_i$, $W_{D_{n+1}}$ is the same but for the additional condition on the group of the change of sign: $\prod_i (\pm)_i = 1$. Since the projection of D_{n+1} on B_n acts as $(e_1, e_2, \ldots, e_n, e_{n+1}) \to (e_1, e_2, \ldots, e_n, 0)$, the singular element $\Phi_{D_{n+1}}^{(\mu)}$ which is the orbit of $W_{D_{n+1}}$ will become a composition of the orbits of W_{B_n} after the projection.

Indeed, if $\mu + \rho_{D_{n+1}} = (a_1, a_2, \ldots, a_{n+1})$ in the standart basis $(e_1, e_2, \ldots, e_{n+1})$, then the Weyl group $W_{D_{n+1}}$ will act on it by permutating a_i and changing their signs. After the projection the last coefficient in every element will be cut. Among these elements will be groups in which all elements will have the same set of a_i standing in arbitrary order and having an arbitrary sign. These groups will be similar to the orbits of the Weyl group W_{B_n} but for the singular miltiplicity. A singular miltiplicity is a determinant $\varepsilon(w)$ of the element w of the Weyl group. Because of the additional condition on the Weyl group $W_{D_{n+1}}$ the change of the signs of a_i doesn't change the singular multiplicity. That's not true for the Weyl group W_{B_n}, thus exactly half of the elements of each newly formed orbit $\tilde{\Phi}_{B_n}^{\tilde{\nu}}$ of W_{B_n} will have the wrong singular multiplicity. Moreover, these orbits can be rewritten in the following form:

$$\tilde{\Phi}_{B_n}^{\tilde{\nu}} = \sum_{w \in W_{D_n}} \varepsilon(w)(1 + s_{e_n}) w e^{\tilde{\nu} + \rho_{B_n}}. \tag{10}$$

This decomposition is possible because the orbit of the Weyl group of D_n coincides with the half of $\tilde{\Phi}_{B_n}^{\tilde{\nu}}$ that has correct singular multiplicities and other half can be obtained by the action of the element s_{e_n} of W_{B_n}.

Thus, the projection $(\Phi_{D_{n+1}}^{(\mu)})'$ consists of $(n+1)$ quasi-orbits (with "wrong" signs) of the Weyl group W_{B_n}. It means that there are $(n+1)$ weights in the fundamental Weyl chamber $\bar{C}_{B_n}$ of B_n. It can be easily shown that these weights are $(a_1, a_2, \ldots, a_n)$ and weights obtained from $(a_1, a_2, \ldots, a_n)$ by consequent subtraction of $\mu_i e_i$ starting with $\mu_n e_n$, where μ_i is Dynkin labels of μ plus 1. As it turns out, if one were to add to this set the required weights with corresponding singular multiplicities to construct the singular element of $\mathfrak{s} = A_1 + A_1 + \cdots + A_1$ as was done in the Introduction of this paper, all additional weight would lie on the boundaries of the fundamental Weyl chamber $\bar{C}_{B_n}$. This fact is easily proved by observing that the action of one of $s_{\alpha_i^{B_n}}$ on the weights

doesn't change those weigths. Thus, the projection $(\Phi^\mu_{D_{n+1}})'$ can be viewed as a quasi-orbit of the singular element of $\mathfrak{s}$:

$$\Phi^{[\mu_1,\mu_2,\dots,\mu_{n+1}]}_{D_{n+1}} = \sum_{w \in W_{D_n}} \varepsilon(w)(1 + s_{e_n})w(e^{\mu'+\rho'_{D_{n+1}} - \tilde{\mu}}\Psi^{[\mu_1,\mu_2,\dots,\mu_n]}_{\mathfrak{s}}), \qquad (11)$$

where $\tilde{\mu} = [\mu_1, \mu_2, \dots, \mu_n]$. This decomposition yields the branching rules similar to (12):

$$b^{[\mu_1,\mu_2,\dots,\mu_{n+1}]}_{(\mu-\phi(\tilde{\mu}-\tilde{\nu}))} = M^{[\mu_1,\mu_2,\dots,\mu_n]}_{(\mathfrak{s})\tilde{\nu}}, \qquad (12)$$

which is in total agreement with the Gelfand-Tsetlin branching rules as the weights of the representations of $\mathfrak{s} = A_1 + A_1 + \dots + A_1$ are always equal to one (see Fig.2 and Fig.3).
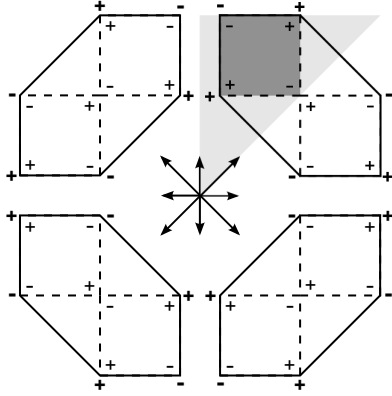

Figure 2: Splint $D_3 = B_2 \cup A_1 + A_1$: projection of the singular element $\Phi^{[1,1,2]}_{D_3}$ can be made from singular elements $\Phi^{[1,1]}_{A_1+A_1}$. Light-grey area is the fundamental Weyl chamber of subalgebra B_2.
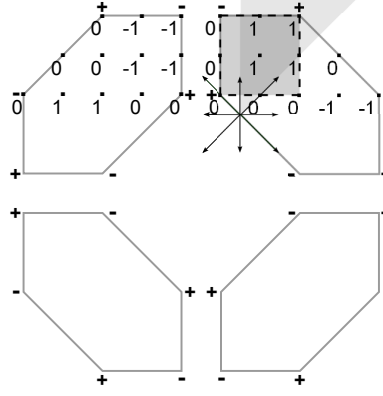
Figure 3: Injection fan applied to the singular element $\Phi^{[1,1]}_{A_1+A_1}$ yields branching coefficients that are equal to 1.

Careful examination of the embedding $G_2 \to B_3$ leads to exactly the same conclusion.

So far we only considered special splints where both embedded systems are root systems. However, it is possible to study the splints where it is not the case. Such splints obviously can not serve to simplify calculation of the branching rules but one may use the properties of special embeddings and the projections to creat a perepesentation theory for the systems that are not root systems.

As the result we see that branching coefficients coincide with weight multiplicities for the splints in the following table:

$$
\begin{array}{cc||c|c}
\text{type} & \Delta & \Delta_{\mathfrak{a}} & \Delta_{\mathfrak{s}} \\
\hline
\text{(i)} & G_2 & A_2 & A_2 \\
 & F_4 & D_4 & D_4 \\
(*) & B_3 & G_2 & A_2 \\
(*) & D_{r+1} & B_r & \oplus^r A_1 \\
\hline
\text{(ii)} & B_r(r \geq 2) & D_r & \oplus^r A_1 \\
\text{(iii)} & A_r(r \geq 2) & A_{r-1} \oplus u\,(1) & \oplus^r A_1 \\
 & B_2 & A_1 \oplus u\,(1) & A_2
\end{array}
\tag{13}
$$

here the splints marked with (*) are special splints.

Conclusion

Computation of branching coefficients is important for different physical models with symmetry breaking. This computation is drastically simplified if branching coefficients coincide with weight multiplicities, since efficient Freudental formula can be used [Moody and Patera, 1982]. Classification of splints for regular and special embeddings gives us all the cases when this coincidence takes place. Aside from computational importance, this coincidence is very interesting from representation-theoretic point of view, because it follows from new unexpected connection between different simple Lie algebras.

There exist special embeddings of semisimple Lie-algebras into simple Lie-algebras. For such embeddings the procedure of finding splints that allow to calculate branching coefficient is similar to the one conducted above. While the results of the procedure are not presented in this paper they can be obtained with little difficulty.

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